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On quadrature for Cauchy principal value integrals of oscillatory functions

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Abstract

The authors develop an algorithm for the numerical evaluation of Cauchy principal value integrals of oscillatory functions. The method is based on an interpolatory procedure at the zeros of the orthogonal polynomials with respect to a Jacobi weight. A numerically stable procedure is obtained and the corresponding algorithm can be implemented in a fast way yielding satisfactory numerical results. Bounds of the error and of the amplification factor are also provided.

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1. Introduction

The numerical evaluation of finite Fourier integrals of the form

$$\int_{-1}^1 e^{i\omega x} f(x) dx, \quad \omega \geq 0, \quad i^2 = -1 \quad (1.1)$$

has wide applications in applied mathematics, physics and engineering. If ω is large the integrand is highly oscillatory and classical methods of integration are unsuitable. The earliest numerical method

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for the treatment of (1.1) is due to Filon [12]. There has been a number of papers dealing with this subject (see, for instance, [1,3,13,14,17,18,24]). Further, the numerical evaluation of some weakly singular oscillatory integrals was already investigated (see [11] and the references cited therein).

In this paper we are concerned with the evaluation of the integral

$$H^\omega(g; t) := \int_{-1}^1 e^{i\omega x} \frac{g(x)}{x-t} dx$$

$$= \lim_{\varepsilon \rightarrow 0^+} \int_{|x-t| \geq \varepsilon} e^{i\omega x} \frac{g(x)}{x-t} dx, \quad -1 < t < 1, \quad (1.2)$$

i.e. the divergence at $x = t$ in (1.2) is allowed for by taking the Cauchy principal value of the integral. The integral (1.2) has two practical difficulties; indeed, it is oscillatory and has a singularity of Cauchy type; for the latter see [4,5,7,8] and their references.

The literature about the numerical approximation of (1.1) and of integrals with a Cauchy singularity is wide and quite satisfactory, while only a small number of publications deals with the numerical evaluation of integrals defined as in (1.2) (see [15,22]). However, the importance coming from its several applications in applied sciences, justifies some interest in its numerical evaluation. Essentially two kinds of quadrature rules of interpolatory type have been proposed to compute integrals with a Cauchy singularity, accordingly as one includes the point t among the quadrature knots or not. Extending the former, Okecha proposed in [15] to compute numerically (1.2) by replacing the function g with the corresponding Lagrange interpolating polynomial on the m zeros $x_{m,k}$, $k=1, \dots, m$, of the m th Legendre polynomial and on the singularity t . In order to give an error bound, the author of [15] made the assumption that $g(x)$ is analytic for $-1 \leq x \leq 1$. On the other hand, since

$$H^\omega(g; t) = \int_{-1}^1 \cos \omega x \frac{g(x)}{x-t} dx + i \int_{-1}^1 \sin \omega x \frac{g(x)}{x-t} dx, \quad -1 < t < 1,$$

we deduce that the existence of $H^\omega(g; t)$ is guaranteed if g is continuous on $[-1, 1]$ and $\int_{-1}^1 \omega(g; \delta) \delta^{-1} d\delta < \infty$, where $\omega(g; \cdot)$ is the ordinary modulus of continuity of g . Furthermore, we know that choosing the points $x_{m,k}$, $k=1, \dots, m$, and t as quadrature knots, some problems can arise in the stability and in the convergence of the related approximation procedure. In fact, if $\omega = 0$ then the quadrature rule proposed in [15] to approximate (1.2) becomes the so-called “Gaussian rule” for the Cauchy principal value integrals. It is well-known that this rule exhibits numerical cancellation when t approaches one of the nodes $x_{m,k}$ (cf. [6,10]). Moreover, in [20] it has been proved that in general such “Gaussian rule” diverges if g is only Hölder continuous. Therefore, the approximation of (1.2) by a quadrature rule of interpolatory type including the point t among the nodes is unsuitable. Indeed, even if the quadrature rule in [15] is stable with respect to the oscillatory factor in (1.2) because it is exactly integrated, the stability and the convergence fail with respect to the singularity t because of the choice of the quadrature knots. On the other hand, in [15] the same author stated that the rule by himself proposed and extensively studied is numerically unstable. To avoid this, Okecha appealed to the idea of leaving the point t out of the quadrature knots (see formula (6.2) in [15]) as already done to compute integrals with a Cauchy singularity [16]. Even if it is stated that the last formula works better than the other one, no convergence and stability result as well as no numerical experiment can be found in [15] to prove its better behaviour.

In the present paper, we refer to the same quadrature formula (6.2) in [15] with a suitable choice of the quadrature knots. In this case we are able to prove the convergence and the stability of the quadrature rule (see Theorem 3.2), giving also an algorithm for its practical implementation. Indeed, in order to approximate the integral (1.2) using rules which require the evaluation of g at nodes independent upon t , we propose a numerical procedure of interpolatory type which uses as quadrature knots the zeros of the Jacobi orthogonal polynomials. We shall describe the algorithm which allows us to compute the quadrature rule in a stable way and an error bound is provided under weak assumptions on the smoothness of g .

2. Quadrature formula

Here we consider formulas of interpolatory type constructed by replacing g by its Lagrange interpolating polynomial based on a set of m distinct nodes. This idea has been already proposed and examined for the evaluation of Cauchy principal value integrals without the oscillatory factor (see [7,8]).

Let

$$v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha, \beta > -1, \quad |x| \leq 1$$

and assume that the quadrature knots coincide with the zeros $x_{m,k}^{\alpha,\beta}$, $k = 1, \dots, m$, of the orthonormal Jacobi polynomial $p_m(v^{\alpha,\beta})$ of degree m associated with $v^{\alpha,\beta}$.

The Lagrange polynomial $\mathcal{L}_m(v^{\alpha,\beta}; g)$ which interpolates the function g at the points $x_{m,k}^{\alpha,\beta}$, $k = 1, \dots, m$, may be written as

$$\mathcal{L}_m(v^{\alpha,\beta}; g; x) = \sum_{k=1}^m \ell_{m,k}(v^{\alpha,\beta}; x) g(x_{m,k}^{\alpha,\beta}), \quad (2.1)$$

where

$$\ell_{m,k}(v^{\alpha,\beta}; x) = \frac{p_m(v^{\alpha,\beta}; x)}{p'_m(v^{\alpha,\beta}; x_{m,k}^{\alpha,\beta})(x - x_{m,k}^{\alpha,\beta})}, \quad k = 1, \dots, m,$$

are the Lagrange fundamental polynomials corresponding to the weight $v^{\alpha,\beta}$. Further, let us denote by $R_m(v^{\alpha,\beta}; g)$ the error due to the Lagrange interpolation, i.e.

$$g(x) = \mathcal{L}_m(v^{\alpha,\beta}; g; x) + R_m(v^{\alpha,\beta}; g; x). \quad (2.2)$$

Now, multiplying (2.2) through by $e^{i\omega x}/(x-t)$ and integrating over $(-1, 1)$, one gets

$$H^\omega(g; t) = H_m^\omega(v^{\alpha,\beta}; g; t) + E_m^\omega(v^{\alpha,\beta}; g; t), \quad (2.3)$$

where

$$H_m^\omega(v^{\alpha,\beta}; g; t) = \sum_{k=1}^m A_{m,k}^\omega(v^{\alpha,\beta}; t) g(x_{m,k}^{\alpha,\beta}), \quad (2.4)$$

$$A_{m,k}^\omega(v^{\alpha,\beta}; t) = H^\omega(\ell_{m,k}(v^{\alpha,\beta}); t), \quad k = 1, \dots, m$$

and

$$E_m^\omega(v^{\alpha,\beta}; g; t) = H^\omega(R_m(v^{\alpha,\beta}; g); t).$$

The quadrature rule (2.3) has degree of exactness at least $m - 1$, i.e. $E_m^\omega(v^{\alpha,\beta}; g) \equiv 0$, whenever g is a polynomial of degree $\leq m - 1$.

We remark that the formula (2.3) is a particular case of (6.2) in [15] having chosen the Jacobi zeros $x_{m,k}^{\alpha,\beta}$, $k = 1, \dots, m$, as quadrature knots. Even if Okecha has proposed the formula (6.2) in [15] in order to give a stable algorithm to compute (1.2), no convergence and stability result can be found in his paper. Our choice of the quadrature knots allows us to go on studying the behaviour of the quadrature rule (2.3).

We can determine a stable representation for the coefficients $A_{m,k}^\omega(v^{\alpha,\beta}; t)$. To this end, we represent the Lagrange interpolating polynomial $\mathcal{L}_m(v^{\alpha,\beta}; g)$ of degree at most $m - 1$ in the form

$$\mathcal{L}_m(v^{\alpha,\beta}; g; x) = \sum_{i=0}^{m-1} a_i^{\alpha,\beta}(g) p_i(v^{\alpha,\beta}; x),$$

where

$$a_i^{\alpha,\beta}(g) = \sum_{k=1}^m \lambda_{m,k}^{\alpha,\beta} p_i(v^{\alpha,\beta}; x_{m,k}^{\alpha,\beta}) g(x_{m,k}^{\alpha,\beta}), \quad i = 0, 1, \dots$$

and $\lambda_{m,k}^{\alpha,\beta}$, $k = 1, \dots, m$, being the Christoffel constants corresponding to the weight $v^{\alpha,\beta}$.

Defining the functions

$$q_m^\omega(v^{\alpha,\beta}; t) = H^\omega(p_m(v^{\alpha,\beta}); t), \quad |t| < 1, \quad m \geq 0, \quad (2.5)$$

formula (2.4) can thus be rewritten in the form

$$\begin{aligned} H_m^\omega(v^{\alpha,\beta}; g; t) &= \sum_{i=0}^{m-1} a_i^{\alpha,\beta}(g) q_i^\omega(v^{\alpha,\beta}; t) \\ &= \sum_{k=1}^m \lambda_{m,k}^{\alpha,\beta} \left[\sum_{i=0}^{m-1} p_i(v^{\alpha,\beta}; x_{m,k}^{\alpha,\beta}) q_i^\omega(v^{\alpha,\beta}; t) \right] g(x_{m,k}^{\alpha,\beta}). \end{aligned} \quad (2.6)$$

From this latter equation we obtain the following representation for the coefficients:

$$A_{m,k}^\omega(v^{\alpha,\beta}; t) = \lambda_{m,k}^{\alpha,\beta} \sum_{i=0}^{m-1} p_i(v^{\alpha,\beta}; x_{m,k}^{\alpha,\beta}) q_i^\omega(v^{\alpha,\beta}; t), \quad k = 1, \dots, m. \quad (2.7)$$

Since the orthonormal polynomials $p_m(v^{\alpha,\beta})$ satisfy a three-term recurrence relation of type

$$\begin{aligned} p_{-1}(v^{\alpha,\beta}; x) &\equiv 0, \\ p_0(v^{\alpha,\beta}; x) &\equiv \left(\int_{-1}^1 v^{\alpha,\beta}(x) dx \right)^{-1/2}, \\ p_{m+1}(v^{\alpha,\beta}; x) &= (A_m x + B_m) p_m(v^{\alpha,\beta}; x) - C_m p_{m-1}(v^{\alpha,\beta}; x), \quad m \geq 0, \end{aligned} \quad (2.8)$$

(see [23]), then also the functions $q_m^\omega(v^{\alpha,\beta})$ defined by (2.5) satisfy a similar relationship

$$\begin{aligned} q_0^\omega(v^{\alpha,\beta}; t) &= p_0(v^{\alpha,\beta}) \int_{-1}^1 \frac{e^{i\omega x}}{x-t} dx, \\ q_1^\omega(v^{\alpha,\beta}; t) &= (A_1 t + B_1) q_0^\omega(v^{\alpha,\beta}; t) + \frac{1}{A_1} M_0^\omega, \\ q_{m+1}^\omega(v^{\alpha,\beta}; t) &= (A_m t + B_m) q_m^\omega(v^{\alpha,\beta}; t) - C_m q_{m-1}^\omega(v^{\alpha,\beta}; t) + A_m M_m^\omega(v^{\alpha,\beta}), \quad m \geq 1, \end{aligned} \quad (2.9)$$

where

$$M_m^\omega(v^{\alpha,\beta}) = \int_{-1}^1 e^{i\omega x} p_m(v^{\alpha,\beta}; x) dx, \quad m \geq 0 \quad (2.10)$$

may be evaluated analytically or can be accurately approximated using any rule designed for oscillatory functions, e.g. [1,18].

In order to compute the starting value $q_0^\omega(v^{\alpha,\beta}; t)$ of (2.9), we remark that

$$\begin{aligned} \Re[q_0^\omega(v^{\alpha,\beta}; t)] &= p_0(v^{\alpha,\beta}) \int_{-1}^1 \frac{\cos \omega x}{x-t} dx \\ &= p_0(v^{\alpha,\beta}) [\cos \omega t [\text{Ci}(\tau_1) - \text{Ci}(|\tau_2|)] - \sin \omega t [\text{Si}(\tau_1) + \text{Si}(|\tau_2|)]] \end{aligned}$$

and

$$\begin{aligned} \Im[q_0^\omega(v^{\alpha,\beta}; t)] &= p_0(v^{\alpha,\beta}) \int_{-1}^1 \frac{\sin \omega x}{x-t} dx \\ &= p_0(v^{\alpha,\beta}) [\sin \omega t [\text{Ci}(\tau_1) - \text{Ci}(|\tau_2|)] + \cos \omega t [\text{Si}(\tau_1) + \text{Si}(|\tau_2|)]], \end{aligned}$$

where

$$\begin{aligned} \text{Ci}(\tau) &= \int_0^\tau \frac{\cos x - 1}{x} dx + \log \tau + C, \quad \tau > 0 \\ \text{Si}(\tau) &= \int_0^\tau \frac{\sin x}{x} dx, \end{aligned}$$

are the sine and cosine integral, respectively; $\tau_1 = \omega(1-t)$, $\tau_2 = -\omega(1+t)$ and C is the Euler constant.

To evaluate $H_m^\omega(v^{\alpha,\beta}; g; t)$ we give an algorithm which uses (2.6), so that we replace the quadrature sum (2.4) involving the coefficients $A_{m,k}^\omega(v^{\alpha,\beta}; t)$ by a linear combination with constant coefficients $a_i^{\alpha,\beta}(g)$ of the values $q_i^\omega(v^{\alpha,\beta}; t)$. The steps of the algorithm are the following:

Algorithm.

1. Compute $a_i^{\alpha,\beta}(g) = \sum_{k=1}^m \lambda_{m,k}^{\alpha,\beta} p_i(v^{\alpha,\beta}; x_{m,k}^{\alpha,\beta}) g(x_{m,k}^{\alpha,\beta})$,
 $i = 0, 1, \dots, m-1$ where $p_i(v^{\alpha,\beta}; x_{m,k}^{\alpha,\beta})$ is computed using (2.8);

2. Compute $H_m^\omega(v^{\alpha,\beta}; g; t) = \sum_{i=0}^{m-1} a_i^{\alpha,\beta}(g) q_i^\omega(v^{\alpha,\beta}; t)$ by means of the following Clenshaw type algorithm:

$$z_{m+1} = z_m = 0, \quad w_m = 0,$$

$$z_k = (A_k t + B_k) z_{k+1} - C_{k+1} z_{k+2} + a_k^{\alpha,\beta}(g), \quad k = m-1, m-2, \dots, 0,$$

$$w_k = A_k M_k^\omega(v^{\alpha,\beta}) z_{k+1} + w_{k+1},$$

$$H_m^\omega(v^{\alpha,\beta}; g; t) = q_0^\omega(v^{\alpha,\beta}; t) z_0 + w_0.$$

An analogous algorithm has been proposed and generalized in [16,8], respectively. In these papers, the authors deal the numerical computation of Cauchy principal value integrals where the integrand is not oscillatory.

We remark that the computation of coefficients $a_i^{\alpha,\beta}(g)$ is not influenced by the value t and oscillatory factor ω , while the computation of the values $q_i^\omega(v^{\alpha,\beta}; t)$ can be done efficiently by means of the recurrence relationship (2.9). Thus the crux of the problem of the implementation is the accurate evaluation of the integrals $M_m^\omega(v^{\alpha,\beta})$. Further, the starting values of (2.9) require the evaluation of the cosine and sine integrals. These latter integrals can be computed by some mathematical software like Mathematica [25].

3. Error analysis

In this section, we consider the uniform convergence for the quadrature rule (2.3).

At first we remark that

$$\Re[H^\omega(g; t)] = \int_{-1}^1 \frac{g(x)}{x-t} \cos \omega x \, dx,$$

$$\Im[H^\omega(g; t)] = \int_{-1}^1 \frac{g(x)}{x-t} \sin \omega x \, dx$$

and

$$\Re[H_m^\omega(v^{\alpha,\beta}; g; t)] = \sum_{k=1}^m \Re[A_{m,k}^\omega(v^{\alpha,\beta}; t)] g(x_{m,k}^{\alpha,\beta}),$$

$$\Im[H_m^\omega(v^{\alpha,\beta}; g; t)] = \sum_{k=1}^m \Im[A_{m,k}^\omega(v^{\alpha,\beta}; t)] g(x_{m,k}^{\alpha,\beta}),$$

whenever g is a real function. Therefore, we shall study the convergence of the sequences $\{\Re[H_m^\omega(v^{\alpha,\beta}; g; t)]\}_{m=1}^\infty$ and $\{\Im[H_m^\omega(v^{\alpha,\beta}; g; t)]\}_{m=1}^\infty$ to $\Re[H^\omega(g; t)]$ and $\Im[H^\omega(g; t)]$, respectively.

If t is a fixed point of $(-1, 1)$ the convergence of $\{H_m^\omega(v^{\alpha,\beta}; g; t)\}_{m=1}^\infty$ to $H^\omega(g; t)$ can be deduced easily from a result due to Rabinowitz [21] on the convergence of interpolatory product rules to

evaluate Cauchy principal value integrals of the form

$$\int_{-1}^1 \frac{f(x)}{x-t} k(x) dx, \quad |t| < 1,$$

where there is no assumption on the sign of k (cf. Theorem 2.4 in [21]). Nevertheless, we are interested to prove the uniform convergence of the proposed quadrature rule (2.4) to $H^\omega(g)$. To this purpose we need some notations.

Let us denote by $\omega_\varphi(f, \delta)$ the modulus of smoothness of a given function f , defined as

$$\omega_\varphi(f; \delta) = \sup_{h \leq \delta} \max_{|x| \leq 1} |\Delta_{h\varphi} f(x)|,$$

where $\varphi(x) = \sqrt{1-x^2}$ and $\Delta_{h\varphi} f(x) = f(x + h/2\varphi(x)) - f(x - h/2\varphi(x))$, (cf. [9]).

Further, we denote by $\|f\|_\infty = \max_{|x| \leq 1} |f(x)|$ the usual uniform norm.

Now we state a theorem showing the behaviour of the functions $\Re[H^\omega(g; t)]$ and $\Im[H^\omega(g; t)]$.

Theorem 3.1. *Let $g \in C^0$ and $\omega \geq 0$. Then, for $|t| < 1$,*

$$\left\{ \begin{array}{l} |\Re[H^\omega(g; t)]| \\ |\Im[H^\omega(g; t)]| \end{array} \right\} \leq C \log \frac{e}{1-t^2} \left\{ (1+\omega) \|g\|_\infty + \int_0^1 \frac{\omega_\varphi(g; \delta)}{\delta} d\delta \right\}, \quad (3.1)$$

where C denotes a positive constant independent of t, g and ω .

Proof. Let $\tilde{G}^\omega(g; x) = g(x) \sin \omega x$ and $\bar{\tilde{G}}^\omega(g; x) = g(x) \cos \omega x$.

Since

$$|\Delta_{h\varphi} \tilde{G}^\omega(g; x)| \leq \left| \sin \omega \left(x + \frac{h}{2} \varphi(x) \right) \right| |\Delta_{h\varphi} g(x)| + \left| g \left(x - \frac{h}{2} \varphi(x) \right) \right| |\Delta_{h\varphi} \sin \omega x|$$

and

$$\sup_{h \leq \delta} \max_{|x| \leq 1} |\Delta_{h\varphi} \sin \omega x| = 2 \sup_{h \leq \delta} \max_{|x| \leq 1} \left| \cos \omega x \sin \omega \frac{h}{2} \varphi(x) \right| \leq \omega \delta,$$

it follows that

$$\omega_\varphi(\tilde{G}^\omega; \delta) \leq \omega_\varphi(g; \delta) + \omega \delta \|g\|_\infty. \quad (3.2)$$

Similarly

$$\omega_\varphi(\bar{\tilde{G}}^\omega; \delta) \leq \omega_\varphi(g; \delta) + \omega \delta \|g\|_\infty. \quad (3.3)$$

By (3.2), (3.3) and Theorem 2.1 in [2], we deduce (3.1). \square

Let us denote by

$$A_m^{\alpha, \beta} = \max_{|x| \leq 1} \sum_{k=1}^m |l_{m,k}(v^{\alpha, \beta}; x)|, \quad m \in \mathbb{N},$$

the m th Lebesgue constant corresponding to the weight $v^{\alpha,\beta}$. Then, for the quadrature rule (2.4) the next result holds true.

Theorem 3.2. For every function $g \in C^0$ and $\omega \geq 0$ we have

$$\left\{ \begin{array}{l} |\Re[H_m^\omega(v^{\alpha,\beta}; g; t)]| \\ |\Im[H_m^\omega(v^{\alpha,\beta}; g; t)]| \end{array} \right\} \leq C \log \frac{e}{1-t^2} (1 + \omega + \log m) A_m^{\alpha,\beta} \|g\|_\infty \quad (3.4)$$

and

$$\left\{ \begin{array}{l} |\Re[E_m^\omega(v^{\alpha,\beta}; g; t)]| \\ |\Im[E_m^\omega(v^{\alpha,\beta}; g; t)]| \end{array} \right\} \leq C \log \frac{e}{1-t^2} (1 + \omega + \log m) A_m^{\alpha,\beta} \int_0^{1/m} \frac{\omega_\phi(g; \delta)}{\delta} d\delta, \quad (3.5)$$

where C denotes a positive constant independent of m, g, ω and $t \in (-1, 1)$.

Proof. In view of (3.1) we can write

$$\begin{aligned} \Re[H_m^\omega(v^{\alpha,\beta}; g; t)] &= \Re[H^\omega(\mathcal{L}_m(v^{\alpha,\beta}; g); t)] \\ &\leq C \log \frac{e}{1-t^2} (1 + \omega) + A_m^{\alpha,\beta} \|g\|_\infty \int_0^1 \frac{\omega_\phi(\mathcal{L}_m(v^{\alpha,\beta}; g); \delta)}{\delta} d\delta. \end{aligned} \quad (3.6)$$

Then, by applying the Bernstein inequality

$$\begin{aligned} &\int_0^1 \frac{\omega_\phi(\mathcal{L}_m(v^{\alpha,\beta}; g); \delta)}{\delta} d\delta \\ &= \left\{ \int_0^{1/m} + \int_{1/m}^1 \right\} \frac{\omega_\phi(\mathcal{L}_m(v^{\alpha,\beta}; g); \delta)}{\delta} d\delta \\ &\leq C \left\{ \frac{1}{m} \|(\mathcal{L}_m'(v^{\alpha,\beta}; g))_\phi\|_\infty + \|(\mathcal{L}_m(v^{\alpha,\beta}; g))\|_\infty \log m \right\} \\ &\leq C \|(\mathcal{L}_m(v^{\alpha,\beta}; g))\|_\infty \{1 + \log m\} \leq C A_m^{\alpha,\beta} \|g\|_\infty \log m. \end{aligned} \quad (3.7)$$

Hence the first inequality in (3.4) follows from (3.6) by using (3.7). Similarly one can prove the other inequality of (3.4).

In the order to prove (3.5) we remark that

$$\begin{aligned} &|\Re[E_m^\omega(v^{\alpha,\beta}; g; t)]| \\ &= |\Re[H^\omega(g - \mathcal{L}_m(v^{\alpha,\beta}; g); t)]| \\ &\leq C \log \frac{e}{1-t^2} \left\{ (1 + \omega) \|g - \mathcal{L}_m(v^{\alpha,\beta}; g)\|_\infty + \int_0^1 \frac{\omega_\phi(g - \mathcal{L}_m(v^{\alpha,\beta}; g); \delta)}{\delta} d\delta \right\}, \end{aligned} \quad (3.8)$$

where we have used (3.1).

Let p be the polynomial of the best approximation of g with degree $(p) \leq m-1$, and $e_{m-1}(g) = \|g - p\|_\infty$. Then we get

$$\|g - \mathcal{L}_m(v^{\alpha,\beta}; g)\|_\infty \leq e_{m-1}(g)\{1 + A_m^{\alpha,\beta}\}. \quad (3.9)$$

On the other hand

$$\begin{aligned} & \int_0^1 \frac{\omega_\varphi(g - \mathcal{L}_m(v^{\alpha,\beta}; g); \delta)}{\delta} d\delta \\ & \leq \int_0^1 \frac{\omega_\varphi(g - p; \delta)}{\delta} d\delta + \int_0^1 \frac{\omega_\varphi(\mathcal{L}_m(v^{\alpha,\beta}; g - p); \delta)}{\delta} d\delta \\ & \leq C \left\{ e_{m-1}(g) \log m + \int_0^{1/m} \frac{\omega_\varphi(g - p; \delta)}{\delta} d\delta + A_m^{\alpha,\beta} \log m e_{m-1}(g) \right. \\ & \quad \left. + \int_0^{1/m} \frac{\omega_\varphi(\mathcal{L}_m(v^{\alpha,\beta}; g - p); \delta)}{\delta} d\delta \right\} \\ & \leq C \left\{ \log m e_{m-1}(g) [1 + A_m^{\alpha,\beta}] + \int_0^{1/m} \frac{\omega_\varphi(g; \delta)}{\delta} d\delta + \frac{1}{m} \|\mathcal{L}'_m(v^{\alpha,\beta}; g - p)\varphi\|_\infty \right\}, \end{aligned}$$

having used Theorem 8.3.1 in [9] in the last inequality.

Thus, we obtain

$$\int_0^1 \frac{\omega_\varphi(g - \mathcal{L}_m(v^{\alpha,\beta}; g); \delta)}{\delta} d\delta \leq C \left\{ \log m e_{m-1}(g) [1 + A_m^{\alpha,\beta}] + \int_0^{1/m} \frac{\omega_\varphi(g; \delta)}{\delta} d\delta \right\}, \quad (3.10)$$

by using the Bernstein inequality.

Combining (3.9) and (3.10) with (3.8) we deduce the first inequality in (3.5). The proof of the other one is analogous. \square

We remark that by inequalities (3.4) we can know the behaviour of the weighted amplification factor. Indeed, we can deduce the following bounds:

$$\left\{ \left\| \Re[H_m^\omega(v^{\alpha,\beta}; g)] \log^{-1} \frac{e}{1-(\cdot)^2} \right\|_\infty, \left\| \Im[H_m^\omega(v^{\alpha,\beta}; g)] \log^{-1} \frac{e}{1-(\cdot)^2} \right\|_\infty \right\} \leq C(1 + \omega + \log m) \|g\|_\infty A_m^{\alpha,\beta}, \quad (3.11)$$

where C denotes a positive constant independent of g and ω .

4. Concluding remark

The bounds (3.4), (3.5) and (3.11) suggest us to compute $H^\omega(g; t)$ numerically by using as quadrature knots the zeros of the Jacobi polynomial $p_m(v^{\alpha,\beta})$ with $\max\{\alpha, \beta\} \leq -\frac{1}{2}$, so that $A_m^{\alpha,\beta} = O(\log m)$.

Further, we have already remarked that the computation of the quadrature sum $H_m^\omega(v^{\alpha,\beta}; g; t)$, $\alpha, \beta > -1$, can be done efficiently by using the algorithm given in Section 2; thus the crux of the implementation is the accurate evaluation of the integral $M_m^\omega(v^{\alpha,\beta})$ defined by (2.11). Here we propose a procedure to evaluate these constants. At first, we observe that from the parity of the functions \cos, \sin and of the orthonormal Jacobi polynomials, we can write

$$\int_{-1}^1 e^{i\omega x} p_m(v^{\alpha,\beta}; x) dx = \begin{cases} \int_{-1}^1 \cos \omega x p_m(v^{\alpha,\beta}; x) dx & \text{if } m = 2n \text{ even } n \geq 1, \\ i \int_{-1}^1 \sin \omega x p_m(v^{\alpha,\beta}; x) dx & \text{if } m = 2n + 1 \text{ odd } n \geq 0. \end{cases}$$

Moreover, since $p_m(v^{\alpha,\beta}; x)$ is a polynomial of degree m , we can use the following expansion in terms of orthonormal Legendre polynomials:

$$p_m(v^{\alpha,\beta}; x) = \sum_{k=0}^m c_k p_k(v^{0,0}; x),$$

where the coefficients c_k can be exactly evaluated by

$$c_k = \int_{-1}^1 p_m(v^{\alpha,\beta}; x) p_k(v^{0,0}; x) dx = \begin{cases} 0 & \text{if } m+k \text{ odd,} \\ \sum_{j=1}^{m+1} \lambda_{m+1,j}^{0,0} p_m(v^{\alpha,\beta}; x_{m+1,j}^{0,0}) p_k(v^{0,0}; x_{m+1,j}^{0,0}) & \text{if } m+k \text{ even.} \end{cases}$$

Moreover, by recalling that (see n.2.17.7.1 in [19])

$$\int_0^a \begin{Bmatrix} \sin bx p_{2n+1}(v^{0,0}; \frac{x}{a}) \\ \cos bx p_{2n}(v^{0,0}; \frac{x}{a}) \end{Bmatrix} dx = (-1)^n \sqrt{\frac{\pi a [2(2n + \delta) + 1]}{b}} \mathcal{J}_{2n+\delta+1/2}, \quad a > 0, \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix},$$

where \mathcal{J}_ν , is the Bessel function of the first kind, we obtain the following expressions:

$$\int_{-1}^1 \cos \omega x p_{2n}(v^{\alpha,\beta}; x) dx = \sum_{\substack{k=0 \\ k \text{ even}}}^{2n} c_k (-1)^{k/2} \sqrt{\frac{\pi(2k+1)}{\omega}} \mathcal{J}_{k+1/2}(\omega),$$

$$\int_{-1}^1 \sin \omega x p_{2n+1}(v^{\alpha,\beta}; x) dx = \sum_{\substack{k=0 \\ k \text{ odd}}}^{2n+1} c_k (-1)^{(k-1)/2} \sqrt{\frac{\pi(2k+1)}{\omega}} \mathcal{J}_{k+1/2}(\omega).$$

Therefore, to evaluate the constants $M_m^\omega(v^{\alpha,\beta})$, in the general case $\alpha, \beta > -1$, we need to use good routines for the evaluation of the Bessel functions of the first kind.

As already observed, the case $\alpha = \beta = -\frac{1}{2}$, is interesting for the convergence and for the stability of the related quadrature rule. Moreover, in this case, the implementation of the proposed method becomes very easy. In fact, when $\alpha = \beta = -\frac{1}{2}$, the zeros and the Christoffel numbers of the Chebyshev polynomials of the first kind are well known, i.e. $\lambda_{m,k}^{-1/2, -1/2} = \frac{\pi}{m}$ and $x_{m,k}^{-1/2, -1/2} = \cos \frac{(2k-1)\pi}{2m}$, $k = 1, \dots, m$. The computation of the quadrature sum $H_m^\omega(v^{-1/2, -1/2}; g; t)$, can be done efficiently by using the algorithm given in Section 2. In order to evaluate $M_m^\omega(v^{-1/2, -1/2})$, starting from the series representation of the Chebyshev polynomials of the first kind, we may write

$$M_m^\omega(v^{-1/2, -1/2}) = \sum_{j=0}^{\left[\frac{m}{2}\right]} c_{m,j} D_{m-2j}^\omega, \quad m = 1, 2, \dots, \quad (4.1)$$

where

$$c_{m,j} = (-1)^j 2^{m-2j-1} \frac{m(m-j-1)!}{j!(m-2j)!}, \quad j = 0, 1, \dots, \left[\frac{m}{2}\right], \quad (4.2)$$

being

$$\left[\frac{m}{2}\right] = \begin{cases} \frac{m}{2} & \text{if } m \text{ is an even integer,} \\ \frac{m-1}{2} & \text{if } m \text{ is an odd integer} \end{cases}$$

and we have set

$$D_k^\omega = \int_{-1}^1 e^{i\omega x} x^k dx, \quad k = 0, 1, 2, \dots$$

Finally, the values D_k^ω , for $k = 2, 3, 4, \dots$, can be computed by the following relation:

$$D_k^\omega = \begin{cases} \frac{2}{\omega} \sin \omega + \frac{2k}{\omega^2} \cos \omega - \frac{k(k-1)}{\omega^2} D_{k-2}^\omega & \text{if } k \text{ is an even integer,} \\ i \left[\frac{2k}{\omega^2} \sin \omega - \frac{2}{\omega} \cos \omega - \frac{k(k-1)}{\omega^2} D_{k-2}^\omega \right] & \text{if } k \text{ is an odd integer,} \end{cases} \quad (4.3)$$

$$D_0^\omega = 2 \frac{\sin \omega}{\omega}, \quad D_1^\omega = 2i \frac{\sin \omega - \omega \cos \omega}{\omega^2}.$$

Thus, $M_m(v^{-1/2, -1/2})$ can be computed for $m = 1, 2, \dots$, and $\omega \geq 0$, by using (4.1) together with (4.2) and (4.3).

5. Numerical examples

The following numerical examples aim to point out that the quadrature rule (2.3) performs better than formulas known at present to approximate integral (1.2) as far as the singularity of Cauchy type as well as the oscillatory factor are concerned. All the computations have been performed in double precision arithmetic.

Table 1
Maximum of the stability factors $K_m^0(v^{0,0}; t)$ and $K_m^{0,0}(t)$

t	$\bar{K}(t)$	$\bar{K}^0(t)$
10^{-1}	1.131131920086713D+01	1.342593780347314D+03
10^{-2}	1.127906877266826D+01	1.806671332230279D+02
10^{-3}	1.133906541364629D+01	1.780648659112735D+03
10^{-4}	1.134931303778011D+01	1.778064685438589D+04
10^{-5}	1.134941558977675D+01	1.777806466750866D+05
10^{-6}	1.134941661530428D+01	1.777780646731887D+06
10^{-7}	1.134941662555956D+01	1.777778065421022D+07
10^{-8}	1.134941662566212D+01	1.777777814022228D+08
10^{-9}	1.134941662566314D+01	1.777777856203502D+09
10^{-10}	1.134941662566314D+01	1.777778533633434D+10
10^{-11}	1.134941662566315D+01	1.77778533523052D+11
10^{-12}	1.134941662566316D+01	1.777853337836735D+12
10^{-13}	1.134941662566315D+01	1.778533667483985D+13

Firstly, we compare (2.3) with the formula

$$H^\omega(g; t) = H_m^{\omega,0}(g; t) + E_m^{\omega,0}(g; t) \quad (5.1)$$

with

$$H_m^{\omega,0}(g; t) = \frac{q_m^\omega(v^{0,0}; t)}{p_m(v^{0,0}; t)} g(t) + \sum_{k=1}^m \frac{(1 - (x_{m,k}^{0,0})^2) q_m^\omega(v^{0,0}; x_{m,k}^{0,0})}{m p_{m-1}(v^{0,0}; x_{m,k}^{0,0}) (x_{m,k}^{0,0} - t)} g(x_{m,k}^{0,0}),$$

proposed by Okecha in [15]. This is a $(m+1)$ -point rule of interpolatory type. The problem arising from the formula by Okecha is due to the fact that it uses the point t among the quadrature knots even if the integral (1.2) is defined as Cauchy principal value integral. For instance, if m is odd and t is very close to 0, then we do not obtain good results by using formula (5.1) since the distance among the quadrature knots t and 0 is very small (see also [6]). On the other hand, the oscillatory factor of (1.2) is exactly integrated by the quadrature rule proposed in [15] as well as by (2.3). To this end, we define the stability factors corresponding to the formulas (2.3) and (5.1) respectively, as

$$K_m^\omega(v^{\alpha,\beta}; t) = \sum_{k=1}^m |A_{m,k}^\omega(v^{\alpha,\beta}; t)|,$$

$$K_m^{\omega,0}(t) = \frac{|q_m^\omega(v^{0,0}; t)|}{|p_m(v^{0,0}; t)|} + \sum_{k=1}^m \frac{(1 - (x_{m,k}^{0,0})^2) |q_m^\omega(v^{0,0}; x_{m,k}^{0,0})|}{m |p_{m-1}(v^{0,0}; x_{m,k}^{0,0})| |x_{m,k}^{0,0} - t|}.$$

We have computed the values of these stability factors assuming $\alpha = \beta = 0$ and $\omega = 0$ since their behaviour does not depend on the oscillatory factor. Setting $\bar{K}(t) = \max_{m=3,4,\dots,128} K_m^0(v^{0,0}; t)$ and $\bar{K}^0(t) = \max_{m=3,4,\dots,128} K_m^{0,0}(t)$, we have obtained the results reported in Table 1. Thus, we deduce that the position of the singularity t does not influence the stability of the formula (2.3) but this is not true for (5.1).

Table 2

 $g(x) = e^x$, $\omega = 0$

m	$E_m(10^{-8})$	$E_m^O(10^{-8})$	$E_m(10^{-10})$	$E_m^O(10^{-10})$	$E_m(10^{-13})$	$E_m^O(10^{-13})$
5	1.16D – 03	1.09D – 08	1.16D – 03	2.02D – 07	1.16D – 03	1.71D – 03
9	2.40D – 08	2.51D – 10	2.40D – 08	6.79D – 07	2.40D – 08	7.32D – 04
17	2.66D – 15	3.47D – 09	2.22D – 15	4.40D – 07	1.77D – 15	2.02D – 07
33	3.02D – 14	5.84D – 08	2.79D – 14	3.22D – 07	3.02D – 14	1.22D – 04
65	1.38D – 14	2.54D – 09	1.78D – 14	9.55D – 08	1.64D – 14	1.22D – 04

Table 3

 $g(x) = \sqrt{1-x^2}$, $\omega = 0$

m	$E_m(10^{-8})$	$E_m^O(10^{-8})$	$E_m(10^{-10})$	$E_m^O(10^{-10})$	$E_m(10^{-13})$	$E_m^O(10^{-13})$
5	3.00D – 10	3.72D – 08	3.00D – 12	3.14D – 10	7.19D – 15	3.14D – 13
9	5.25D – 11	2.11D – 09	5.12D – 13	3.14D – 10	1.09D – 14	4.88D – 4
17	6.94D – 12	1.61D – 09	7.34D – 14	2.39D – 07	4.68D – 15	0.00D+00
33	8.05D – 13	5.84D – 09	2.38D – 14	4.77D – 07	1.56D – 14	1.22D – 04
65	5.43D – 14	2.54D – 09	3.22D – 14	2.98D – 07	3.27D – 15	6.10D – 05

We have compared numerically the rules (2.3) and (5.1) assuming again $\alpha = \beta = 0$ and $\omega = 0$. For increasing values of $m \in \mathbb{N}$ and in various points t , in Tables 2 and 3 we report some numerical results obtained with respect to the test functions $g(x) = e^x$ and $g(x) = \sqrt{1-x^2}$. Denoting by $E_m(t)$ and $E_m^O(t)$ the absolute errors obtained using (2.3) and (5.1), respectively, we have compared them with the solution computed by the Mathematica package in the first example and with the exact solution $-\pi t$ in the other case.

We also evaluate the integral $H^\omega(g; t)$ with the so-called “product quadrature” rule for Cauchy principal value integrals, using the weight function $w(x) = 1$ which corresponds to the Legendre product quadrature rule and we denote this by $\Phi_m^\omega(v^{0,0}; g; t)$, i.e.

$$\Phi_m^\omega(v^{0,0}; g; t) = \int_{-1}^1 \frac{\mathcal{L}_m(v^{0,0}; f; x)}{x-t} dx, \quad f(x) = e^{i\omega x} g(x). \quad (5.2)$$

In other words, the previous formula is obtained with the same procedure used here, but with $\alpha = \beta = 0$ and $e^{i\omega x} g(x)$ in place of $g(x)$ (see formulae (2.1)–(2.4)). Such kind of formula is stable with respect to the singularity t since it does not include t among the quadrature knots [6]. Nevertheless, even if Theorems 3.2 and 2.1 in [2], show the same rate of convergence for the two methods in comparison, as we can see by the tables, the product quadrature method becomes unstable as the value of ω increases, while the method proposed here, does not show this problem. On the other hand, the bad behaviour of the formula (5.2) is expected, since the oscillatory nature of the problem is forced into the function to be interpolate.

For sake of simplicity in all the tables, we denote the real part of the solutions obtained with the two different methods by $I_m^1(t) = |\Re[H_m^\omega(v^{-1/2, -1/2}; g, t)]|$ and $I_m^2(t) = |\Re[\Phi_m^\omega(v^{0,0}; g, t)]|$, respectively.

Table 4

 $g(x) = e^x$, $\omega = 50$

m	$E_m^1(0.0)$	$E_m^2(0.0)$	$E_m^1(0.1)$	$E_m^2(0.1)$	$E_m^1(0.5)$	$E_m^2(0.5)$	$E_m^1(0.9)$	$E_m^2(0.9)$
4	0.2D – 05	0.1D + 00	0.1D – 01	0.2D + 01	0.1D – 02	0.2D + 01	0.3D – 02	0.7D + 01
8	0.8D – 09	0.8D + 00	0.4D – 06	0.2D + 01	0.3D – 07	0.5D + 00	0.5D – 06	0.7D + 01
16	0.4D – 15	0.4D + 00	0.3D – 14	0.1D + 01	0.4D – 15	0.4D + 01	0.1D – 13	0.3D + 01
32	0.6D – 15	0.2D – 03	0.8D – 15	0.1D + 00	0.1D – 15	0.4D + 00	0.1D – 13	0.1D + 01

Table 5

 $g(x) = e^x$, $\omega = 100$

m	$E_m^1(0.0)$	$E_m^2(0.0)$	$E_m^1(0.1)$	$E_m^2(0.1)$	$E_m^1(0.5)$	$E_m^2(0.5)$	$E_m^1(0.9)$	$E_m^2(0.9)$
4	0.7D – 05	0.1D + 01	0.8D – 02	0.2D + 01	0.2D – 02	0.1D + 00	0.3D – 02	0.1D + 02
8	0.4D – 10	0.4D – 01	0.2D – 06	0.3D + 01	0.8D – 07	0.5D + 01	0.5D – 07	0.8D + 01
16	0.1D – 15	0.2D + 00	0.8D – 15	0.1D + 01	0.1D – 14	0.4D + 01	0.1D – 13	0.9D + 01
32	0.4D – 15	0.3D + 00	0.1D – 14	0.3D + 01	0.2D – 15	0.3D + 01	0.2D – 13	0.1D + 02

Table 6

 $g(x) = \sqrt{1-x^2}$, $\omega = 10$

m	$I_m^1(0.1)$	$I_m^2(0.1)$	$I_m^1(0.5)$	$I_m^2(0.5)$	$I_m^1(0.9)$	$I_m^2(0.9)$
4	–0.2D + 01		0.2D + 01			
8	–0.26D + 01		0.263D + 01		–0.3D + 00	
16	–0.262D + 01	–0.26D + 01	0.263D + 01	0.26D + 01	–0.3D + 00	–0.3D + 00
≥ 32	–0.26269D + 01	–0.26269D + 01	0.2633D + 01	0.2633D + 01	–0.38D + 00	–0.38D + 00

Moreover, when we can compare the obtained results with the exact solution, we denote by $E_m^1(t) = |\Re[H^\omega(g, t) - H_m^\omega(v^{-1/2, -1/2}; g, t)]|$ and $E_m^2(t) = |\Re[H^\omega(g, t) - \Phi_m^\omega(v^{0,0}; g, t)]|$, the real part of the errors obtained with the two different methods.

We have chosen the test function $g(x) = e^x$. In Tables 4 and 5 we show the errors of the two methods, compared with the exact solution of the integral evaluated with the Mathematica package, in various points t of the interval of integration $(-1, 1)$, for various values of ω and increasing values of $m \in \mathbb{N}$.

In this last example we have chosen the test function $g(x) = \sqrt{1-x^2}$. Since in this case we are not able to know the exact solution, in Tables 6 and 7 we show the value of the integral obtained using the two different methods, as usual in various points t of the interval of integration $(-1, 1)$, for various values of ω and for increasing values of $m \in \mathbb{N}$. In the following tables we mention only the cases when we obtain correct digits, showing that in order to obtain this the product formula requires a number m of knots larger than one needed to formula (2.3).

Table 7

$$g(x) = \sqrt{1-x^2}, \omega = 100$$

m	$I_m^1(0.1)$	$I_m^2(0.1)$	$I_m^1(0.5)$	$I_m^2(0.5)$	$I_m^1(0.9)$	$I_m^2(0.9)$
4	0.17D+01				−0.1D+01	
8	0.170D+01		0.71D+00		−0.1D + 01	
16	0.1700D + 01		0.712D + 00		−0.123D + 01	
32	0.17002D + 01		0.712D + 00		−0.1235D + 01	
64	0.17002D + 01		0.71221D + 00		−0.12357D + 01	−0.12D + 01
≥ 128	0.17002D + 01	0.17002D + 01	0.71221D + 00	0.71221D + 00	−0.12357D + 01	−0.12357D + 01

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